The Product and Quotient Rule

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Agenda

1. The Product Rule Definition
2. The Product Rule Examples
3. The Quotient Rule Definition
4. The Quotient Rule Examples

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Reason for the Product Rule

The Product Rule must be utilized when the derivative of the product of two functions is to be taken.

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The Product Rule

If *f* and *g* are both differentiable, then:

$$\frac{d}{dx}\left(f(x) * g(x)\right) = \frac{d}{dx}\left(f(x)\right) * g(x) + f(x) * \frac{d}{dx}\left(g(x)\right)$$

which can also be expressed as:

$$\left(f(x) * g(x)\right)' = f'(x) * g(x) + f(x) * g'(x)$$

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The Product Rule in Words

The Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Examples of the Product Rule

Example 1:

If $f(x) = xe^x$, find $f'(x)$

By the Product Rule we have:

$$f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x+1)e^x$$

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Examples of the Product Rule Cont.

Example 2:

Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$

By the Product Rule we have:

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}\left(\sqrt{t}\right)$$

$$= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

Examples of the Product Rule Cont.

Example 3:

$f(x) = [\sin(x)][\cos(x)]$

With the use of the Product Rule the derivative is:

$$f'(x) = \frac{d}{dx}[\sin(x)] * \cos(x) + [\sin(x)] * \frac{d}{dx}[\cos(x)]$$

$$= [\cos(x)][\cos(x)] + [\sin(x)][-\sin(x)]$$

$$= \cos^2(x) - \sin^2(x)$$

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Reason for the Quotient Rule

The Product Rule must be utilized when the derivative of the quotient of two functions is to be taken.

The Quotient Rule

If f and g are both differentiable, then:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}\left(f(x)\right) * g(x) + f(x) * \frac{d}{dx}\left(g(x)\right)}{[g(x)]^2}$$

which can also be expressed as:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) * g(x) + f(x) * g'(x)}{[g(x)]^2}$$

The Quotient Rule in Words

The Quotient Rule says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Examples of the Quotient Rule

Example 1:

$$\text{Let } y = \frac{x^2 + x - 2}{x^3 + 6}$$

$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

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Examples of the Quotient Rule Cont.

Example 2:

If

$$y = e^x/(1+x^2)$$

Then with the use of the Quotient Rule the derivative is:

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$$

Examples of the Quotient Rule Cont.

Example 3:

$f(x) = \tan(x)$

With the use of the Quotient Rule the derivative is:

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$f'(x) = \frac{\frac{d}{dx}[\sin(x)] * \cos(x) + [\sin(x)] * \frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$

$$= \frac{[\cos(x)][\cos(x)] + [\sin(x)][-\sin(x)]}{\cos^2(x)}$$

$$= \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)} = 1 - \tan^2(x) = \sec^2(x)$$

References

- Stewart, James. *Calculus*. Belmont, CA: Thomson Brooks/Cole, 2008.