Trigonometric Substitution

A Tool for Evaluating Integrals

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In This Presentation…

- We will identify keys to determining whether or not to use trig substitution
- Learn to use the proper substitutions for the integrand and the derivative
- Solve the integral after the appropriate substitutions

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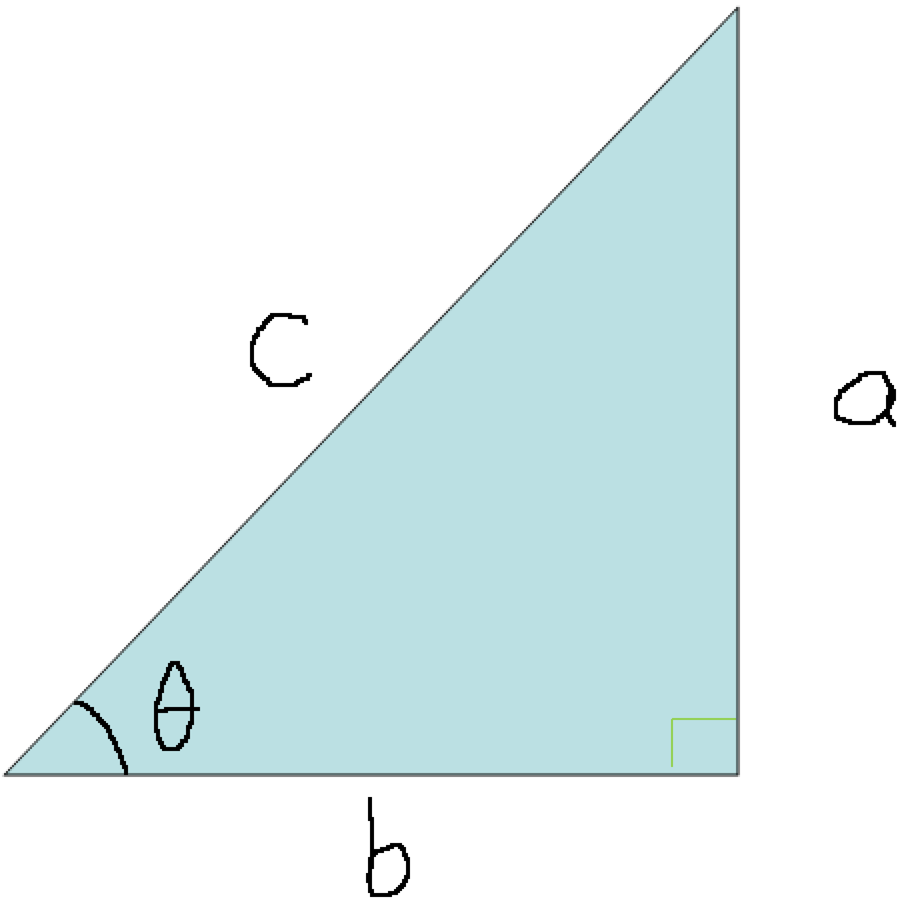
Background

- Based on Pythagorean Theorem:
- $a^2+b^2=c^2$
- $\sin\theta=a/c$
- $\cos\theta=b/c$
- $\tan\theta=a/b$



Background

- Usually, one side is unknown, and is written in terms of the other two sides:

$$c = \sqrt{a^2 + b^2}$$

$$a = \sqrt{c^2 - b^2}$$

$$b = \sqrt{c^2 - a^2}$$

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Steps to Solve

- First we identify if we need trig substitution to solve the problem
- We see if there is a part of the integrand that resembles one of the variations of the Pythagorean Theorem

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Identification

- For example:

$$\int \frac{1}{x^2+4}dx$$

Resembles $1/(a^2+b^2)$

$$\int \frac{1}{\sqrt{9-x^2}}dx$$

Resembles $1/(c^2-a^2)^{1/2}$

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Identification

$$\int \frac{1}{x^2} dx$$

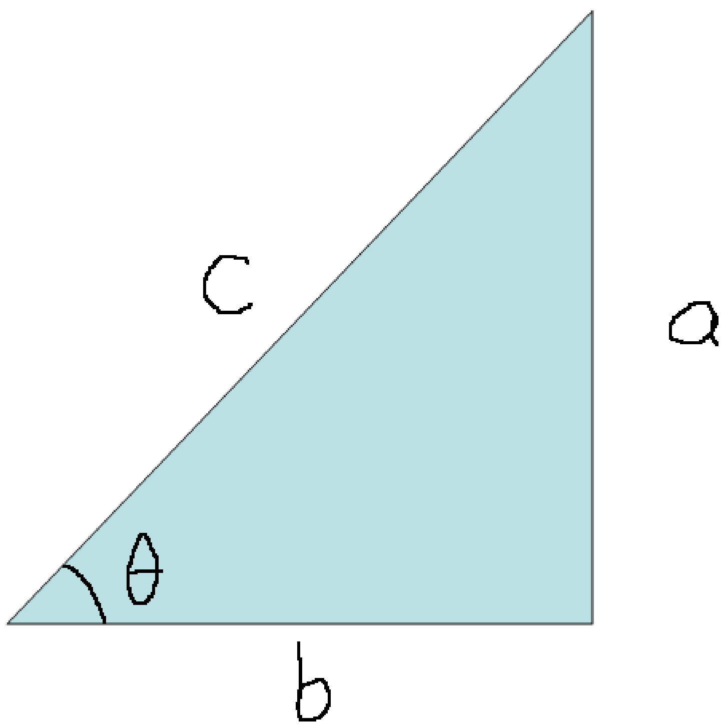
Does not resemble any of the forms

Translation

- Now that we have identified when to use trig substitution, we apply it by changing the coordinate from x to θ, using the triangle.

- ALWAYS DRAW A TRIANGLE WHEN PERFORMING TRIG SUBSTITUTION!!

Translation



- We will use the following example:

$$\int \frac{1}{\sqrt{1-x^2}}dx$$

- Note that it has the form $1/(c^2-a^2)^{1/2}$, where c is $1^{1/2}=1$ and a is $(x^2)^{1/2}=x$

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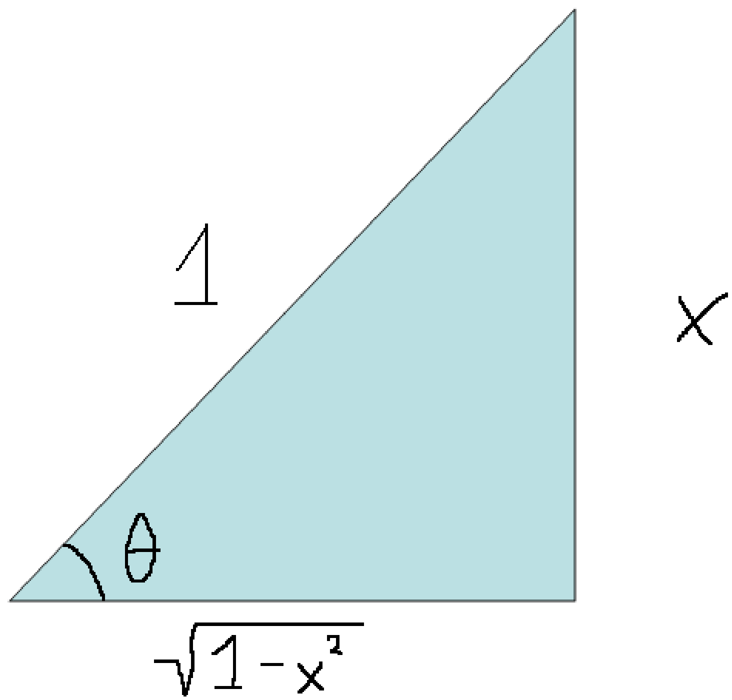
Translation

- First we replace the sides of the general triangle we have with the values that we know, in this case 1 and x.

- Note that by Pythagorean Theorem the second leg becomes $(c^2-a^2)^{1/2} = (1-x^2)^{1/2}$, which is the integrand

Translation

- Note: it is always a good idea to put the x value directly across from θ, as shown



Translation

- Let's begin by finding an equation for x in terms of θ. This is where placing the x across from the θ is useful. Since c is equal to 1, the translation is simply:

$$x=\sin\theta$$

Derivative

- Now that we have an equation for x, we can find an equivalent equation for the derivative, dx, in the integral
- Note that dx can NEVER be ignored, because an integral cannot be solved without a derivative involved

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Derivative

- For our translation of x, the derivative becomes:

$$dx = \cos\theta \, d\theta$$

- Do not forget the $d\theta$ that is obtained from the translated derivative.

Substitution

- Note that the problem can now be solved by substituting x and dx into the integral; however, there is a simpler method.

- If we find a translation of θ that involves the $(1-x^2)^{1/2}$ term, the integral changes into an easier one to work with

Translation Again

- The easier translation to take is to correlate the $(1-x^2)^{1/2}$ term and the c value of 1
- Check to make sure that you understand that the translation is simply:

$$\cos\theta = \sqrt{1-x^2}$$

Substitution

- Now, when we substitute we obtain:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \longrightarrow \int \frac{1}{(\cos\theta)}(\cos\theta\, d\theta)$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

- Which simply becomes:

$$\int d\theta$$

Integration (Finally!)

- After integrating, we obtain:

$$\int d\theta \longrightarrow \theta + C$$

- Now if we back substitute we finally obtain:

Final Answer

$$\sin^{-1} x + C \longleftarrow$$

More Examples

- Let's solve the following integral:

$$\int_0^{4/5} \frac{1}{16+25x^2}\,dx$$

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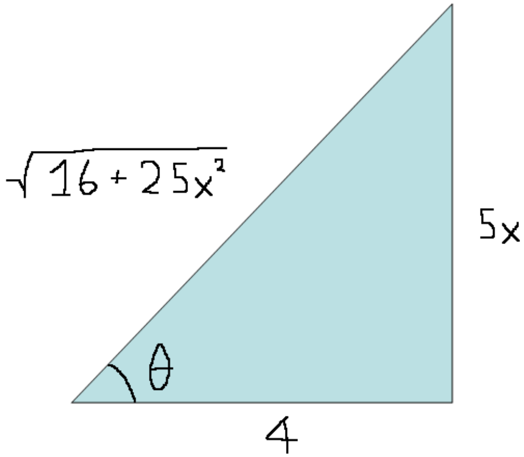

Example 1

- Now we choose a proper translation from x to θ. Show that the translation is:

$$x=4/5*\tan\theta$$

- And the derivative becomes:

$$dx=4/5*\sec^2\theta d\theta$$

- And the substitution that involves the integrand becomes:

$\cos\theta=4/(16+25x^2)^{1/2}$ $16+25x^2=16\sec^2\theta$

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Example 1

- With the substitutions:

$$\int_0^{4/5} \frac{1}{16+25x^2} \longrightarrow \int \frac{1}{(16\sec^2\theta)}(\frac{4}{5}\sec^2\theta d\theta)$$

- Which becomes:

$$\int \frac{1}{20}\theta d\theta)$$

Example 1

- Show that after integration and back substitution we obtain:

$$\left.\frac{1}{20}\tan^{-1}\left(\frac{5x}{4}\right)\right]_{0}^{4/5}$$

- Remember there is no constant this time. After evaluation:

  Answer = π/80

Example 2

- Let’s solve the following integral:

$$\int \sqrt{81 - x^2}\, dx$$

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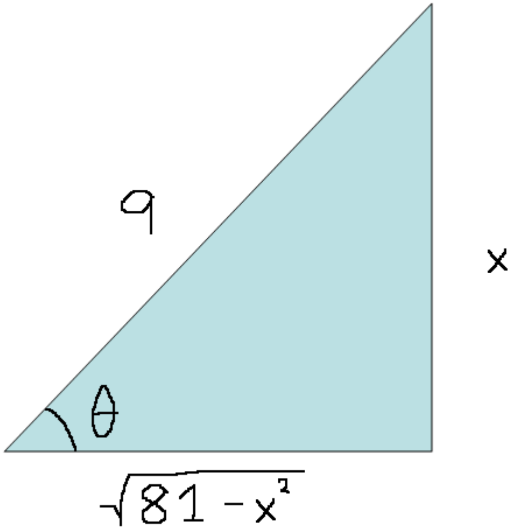

Example 2

- Now let's choose a proper translation from x to θ. Show that the translation is:

$$x=9*\sin\theta$$

- And the derivative becomes:

$$dx=9*\cos\theta d\theta$$

- And the substitution that involves the integrand becomes:

$\cos\theta=(81-x^2)^{1/2}/9$ $\longrightarrow$ $(81-x^2)^{1/2}=9*\cos\theta$

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Example 2

- With the substitutions:

$$\int \sqrt{81 - x^2}\,dx \longrightarrow \int (9\cos\theta)(9\cos\theta\, d\theta)$$

- Which becomes:

$$\int 81\cos^2\theta\, d\theta$$

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Example 2

- For this particular integral, trig substitution is not enough
- Show that the proper trig identity to be used changes the integral to:

$$\int 81\cos^2\theta d\theta \longrightarrow \int 81\left(\frac{1}{2}+\frac{\cos 2\theta}{2}\right)d\theta$$

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Example 2

- Show that after integration we obtain:

$$81(\frac{1}{2}\theta+\frac{\sin 2\theta}{4})+C$$

- Recall that sin2θ=2sinθ*cosθ and so we have:

$$81(\frac{1}{2}\theta+\frac{\sin\theta\cos\theta}{2})+C$$

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Example 2

- And so after back substitution we have, as a final answer:

$$\frac{81}{2}\left(\sin^{-1}\left(\frac{x}{9}\right)+\left(\frac{x}{9}\right)\left(\frac{\sqrt{81-x^2}}{9}\right)\right)+C$$

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Summary

- This is the basic procedure for solving integrals that require trig substitution
- Remember to ALWAYS draw a triangle to help with the visualization process and to find the easiest substitutions to use

References

- Calculus – Stewart 6th Edition
  - Section 6.6 “Inverse Trigonometric Functions”
  - Section 7.3 “Trigonometric Substitution”
  - Appendixes A1, D “Trigonometry”

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