Macaulay Duration

ARC Workshop for BUS
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Macaulay Duration

• What is the Macaulay Duration?
• The Macaulay Duration, Dm, of a collection of cash flows, CFj, is a weighted average (mean) of times (periods), j, at which the cash flows occur, where the weights are the percent of the present value of the cash flow with respect to the sum of the present value of all the cash flows (the value of the cash flow at time 0). It can be thought of as the average economic life time (balance point) of a collection of cash flows.
Macaulay Duration

• Formula:

\[ D_M = \sum_{j=1}^{n} \left[ \frac{CF_j}{(1+i)^j} \right] \times j = \frac{\sum_{j=1}^{n} \frac{CF_j}{(1+i)^j} \times j}{\sum_{j=1}^{n} \frac{CF_j}{(1+i)^j}} = \frac{CF_1 \times 1 + CF_2 \times 2 + \ldots + CF_n \times n}{P} \]

where

\[ P = \sum_{j=1}^{\infty} \frac{CF_j}{(1+i)^j} \]

is the value of the cash flow at time 0.
Macaulay Duration

• Indeed the Macaulay Duration is a measure of the elasticity of the price of the cash flow versus the periodic yield to maturity, i.e. That is it relates the percent change in the price of a cash flow to the percent change in the yield to maturity. The elasticity is best discussed in an environment involving calculus.
Macaulay Duration

• We find a closed form for using pre-calculus methods. We consider a bond in which the face value and redemption value are equal, i.e. $F = M$.

\[
P = \frac{C}{(1+i)^n} + \frac{C}{(1+i)^2} + ... + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n} = C \times a_{n, i} + \frac{F}{(1+i)^n}.
\]

Claim: $S = \sum_{j=1}^{n} \frac{C}{(1+i)^j} \times j = \frac{(1+i)C}{i^2} \left[1 - \frac{1}{(1+i)^n}\right] - \frac{nC}{i(1+i)^n}$

Proof:

\[
S = \frac{C \times 1}{(1+i)^2} + \frac{C \times 2}{(1+i)^3} + ... + \frac{C \times n}{(1+i)^n} = \frac{C \times 1}{(1+i)^2} + \frac{C \times 2}{(1+i)^3} + ... + \frac{C \times n}{(1+i)^n+1}
\]

\[
S - \frac{S}{1+i} = S \left[1 - \frac{1}{1+i}\right] = S \left[\frac{i}{1+i}\right] = \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + ... + \frac{C}{(1+i)^n} - \frac{nC}{(1+i)^{n+1}}
\]

\[
S = \frac{1+i}{i} \left\{ \frac{C}{(1+i)^2} + ... + \frac{C}{(1+i)^n} - \frac{nC}{(1+i)^{n+1}} \right\} = \frac{1+i}{i} \left\{ C \times a_{n, i} - \frac{nC}{(1+i)^{n+1}} \right\}
\]

\[
S = \frac{1+i}{i} \left\{ \frac{1 - \frac{1}{(1+i)^n}}{i} \right\} - \frac{nC}{i(1+i)^n} = \frac{(1+i)C}{i^2} \left[1 - \frac{1}{(1+i)^n}\right] - \frac{nC}{i(1+i)^n}
\]
Macaulay Duration

Now from the definition of Macaulay Duration

\[
D_M = \frac{S + \frac{F \times n}{(1+i)^n}}{P} = \frac{(1+i)C}{i} \left[ 1 - \frac{1}{(1+i)^n} \right] - \frac{nC}{i(1+i)^n} + \frac{F \times n}{(1+i)^n}
\]

\[
D_M = \frac{C}{i} \left[ 1 - \frac{1}{(1+i)^n} \right] + \frac{F}{(1+i)^n}
\]

\[
D_M = D_M \times \frac{i(1+i)^n}{i(1+i)^n} = \frac{(1+i)C}{i} \left[ (1+i)^n - 1 \right] - nC + iFn
\]

Now add and subtract the quantity \((1+i)F\) to the numerator of the last expression. Clearly the quotient is unchanged and we have

\[
D_M = \frac{(1+i)C}{i} \left[ (1+i)^n - 1 \right] + iFn - (1+i)F - nC + iFn
\]

\[
D_M = \frac{1+i}{i} \left\{ C[(1+i)^n - 1] + iF \right\} - (1+i)F - nC + iFn
\]

\[
D_M = \frac{1+i}{i} \left( (1+i)F + nC - iFn \right)
\]

\[
D_M = \frac{(1+i)F + nC - iFn}{C[(1+i)^n - 1] + iF}
\]

Continued...
Macaulay Duration

• Now multiply the numerator and denominator of the second addend by 1/F.

\[ D_M = \frac{1+i}{i} - \frac{(1+i) + n\left(\frac{C}{F}\right) - in}{\frac{C}{F}[(1+i)^n - 1] + i} \]

• Now the periodic (semiannual) coupon rate is. So

• Rule:

\[ D_M = \frac{1+i}{i} - \frac{(1+i) + n(r-i)}{r[(1+i)^n - 1] + i} \]
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Example: Consider a 2-year coupon bond with a face and redemption value of $100 and a coupon rate of 10% per annum payable semiannually and a yield to maturity of 12% per annum compounded semiannually. Find the Macaulay Duration.

Solution: $n = 4, r = .05, i = .06, F = M = 100$

\[ D_M = \frac{1 + .06}{.06} - \frac{(1 + .06) + 4(.05 - .06)}{.05[(1.06)^4 - 1] + .06} = 17.6667 - \frac{1.02}{.05[.2625 - 1] + .05} = 17.6667 - \frac{1.02}{.0731} \]

\[ D_M = 17.6667 - 13.9535 = 3.7132 \]

The Macaulay Duration is 3.7132 semiannual periods or 1.86 years.
Macaulay Duration

• Why do we need the Duration?
All other things being equal, the longer (shorter) the term of the bond, n, the greater (lesser) the magnitude of the percent change. Term and percent change are directly related. So to get a more sensitive bond one can (all other things being equal) use a longer term bond.

However all other things are usually not equal. It is the case that the lesser (greater) the coupon rate the greater (lesser) the percent change for a given change in rates. So coupon rate and percent change are inversely related.

So just changing term or coupon rate does not guarantee the desired change in price, i.e. does not necessarily give the appropriate sensitivity. There is a tool that takes into account the term, coupon rate and yield and is directly related to percent change. It is the Duration. Increase Duration implies greater percent change. Decrease Duration implies lesser percent change.

The units of Duration are time units (periods or years). Duration is the bond manager’s tool for structuring a portfolio of bonds to have the desired sensitivity.
Macaulay Duration

• A balance point

Consider a collection of 10 numbers as follows:

2, 1, 4, 2, 2, 3, 5, 3, 1, 2

So, there are two 1s, four 2s, two 3s, one 4, and one 5. A graph of frequency (how many) versus number looks like

[Bar graph with bars for 1, 2, 3, 4, 5 with counts of 1, 4, 2, 1, 1 respectively]
**Definition:** Torque is the tendency of a force (such as gravity) to rotate an object about a pivot point. A torque can be thought of as a twist or turning force.

There is one point (number) along the number line where the torques cancel – the balance point. That point or number is where all of the weight can be thought to be concentrated. What is it? What is $x$ bar?

$$2 \times (1 - \bar{x}) + 4 \times (2 - \bar{x}) + 2 \times (3 - \bar{x}) + 1 \times (4 - \bar{x}) + 1 \times (5 - \bar{x}) = 0$$

So,

$$2 \times 1 + 4 \times 2 + 2 \times 3 + 1 \times 4 + 1 \times 5 - [2 + 4 + 2 + 1 + 1] \times \bar{x} = 0$$

$$25 - 10 \times \bar{x} = 0$$

$$\bar{x} = 2.5$$

*Continued...*
Macaulay Duration

2.5 is called the balance point or center of gravity or center of mass. It is also the weighted mean of the collection of ten numbers. Observe that

\[
2.5 = \frac{2 \times 1 + 4 \times 2 + 2 \times 3 + 1 \times 4 + 1 \times 5}{10} = .2 \times 1 + .4 \times 2 + .2 \times 3 + .1 \times 4 + .1 \times 5
\]

Each distinct number or class in the collection is multiplied by its relative frequency (relative weight) to result in the balance point or weighted mean. Observe that the weights are numbers between 0 and 1 and have a sum of 1 ( ). Indeed the rule that we have demonstrated is
Macaulay Duration

**Rule:** A collection of numbers multiplied by numbers (relative weights) between 0 and 1 which add up to 1 is the balance point of the collection in the sense that the sum of the torques is 0.

**Series 1**

```

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<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
```

Series 1
Macaulay Duration

Now consider the Macaulay Duration of a bond. By definition

\[ D_M = \left( \frac{PVCF_1}{P} \right) \times 1 + \left( \frac{PVCF_2}{P} \right) \times 2 + \ldots + \left( \frac{PVCF_n}{P} \right) \times n \]

Each present value of cash flow \( j \) divided by \( P \) (which is the price of the bond, i.e. the sum of the present values of all of the cash flows) is a number between 0 and 1 which sum to 1. Hence the Macaulay Duration is the balance point of the collection \{1, 2, 3, ..., n\}, the time periods at which the cash flows occur. It is the average economic lifetime of a cash flow adjusted for the time value of money.
Macaulay Duration
Macaulay Duration

**Rule:** As $n$ (term) increases (decreases) the balance point is pulled to the right (left); that is increases (decreases). Term and Macaulay Duration are directly related.

**Rule:** As $C$ (or coupon rate) increases (the bars get higher) the balance point is pulled to the left (remember that maturity value remains constant); that is decreases. As $C$ decreases the bars get lower and the balance point is pulled to the right; that is increases. Coupon rate and Macaulay Duration are inversely related.

So by viewing the Macaulay Duration as a balance point of periods it becomes clear how Macaulay Duration is related to term (directly) and coupon rate (inversely).
Have Fun with Macaulay Duration

Thank you very much!