Linear optimization

Academic Resource Center
Decision making problem

- linear programing is an important branch of decision making problems modeled and those of optimizing a linear function of decision variables subject to linear constraints that may include equality constraints, inequality constraints and bounds in decision variables.
Decision making problem 1

- A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped each day. If each scientific calculator sold results in a $2 loss, but each graphing calculator produces a $5 profit, how many of each type should be made daily to maximize net profits?
DECISION MAKING PROBLEM 1 (ANALYSIS)

The question asks for the optimal number of calculators, so my variables will stand for that:

\[ x: \text{number of scientific calculators produced} \]
\[ y: \text{number of graphing calculators produced} \]

Since they can't produce negative numbers of calculators, I have the two constraints, \( x \geq 0 \) and \( y \geq 0 \). But in this case, I can ignore these constraints, because I already have that \( x \geq 100 \) and \( y \geq 80 \). The exercise also gives maximums: \( x \leq 200 \) and \( y \leq 170 \). The minimum shipping requirement gives me \( x + y \geq 200 \); in other words, \( y \geq -x + 200 \). The revenue relation will be my optimization equation: \[ R = -2x + 5y. \]
Decision MAKING PROBLEM 1 (ANSWER)

- \( R = -2x + 5y \), subject to:
  \[ 100 < x < 200 \]
  \[ 80 < y < 170 \]
  \[ y > -x + 200 \]

- When you test the corner points at (100, 170), (200, 170), (200, 80), (120, 80), and (100, 100), you should obtain the maximum value of \( R = 650 \) at \((x, y) = (100, 170)\). That is, the solution is "100 scientific calculators and 170 graphing calculators".
Decision making problem 2

- You need to buy some filing cabinets. You know that Cabinet X costs $10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs $20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given $140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume?
The question ask for the number of cabinets I need to buy, so my variables will stand for that:

- \( x \): number of model X cabinets purchased
- \( y \): number of model Y cabinets purchased

Naturally, \( x > 0 \) and \( y > 0 \). I have to consider costs and floor space (the "footprint" of each unit), while maximizing the storage volume, so costs and floor space will be my constraints, while volume will be my optimization equation.
Decision MAKING PROBLEM 2

- cost: \( 10x + 20y < 140 \), or \( y < -\frac{1}{2}x + 7 \)
- space: \( 6x + 8y < 72 \), or \( y < -\frac{3}{4}x + 9 \)
- volume: \( V = 8x + 12y \)
- When you test the corner points at (8, 3), (0,7), and (12, 0), you should obtain a maximal volume of 100 cubic feet by buying eight of model X and three of model Y.
Optimal Path Problems

- Shortest Path
- Maximum Flow
- Critical Path
Shortest path problem

- The problem is to determine the best way to traverse a network to get from an origin to a given destination as cheaply as possible. Suppose that in a given network there are $m$ nodes and $n$ arcs (i.e. edges) and a cost $C_{ij}$ associated with each arc $(i \to j)$ in the network. Formally, the Shortest Path (SP) problem is to find the shortest (least cost) path from the start node 1 to the finish node $m$. The cost of the path is the sum of the costs on the arcs in the path. Define binary variables $X_{ij}$, where $X_{ij} = 1$ if the arc $(i \to j)$ is on the SP and $X_{ij} = 0$ otherwise. There are two special nodes, called the origin and destination. The objective is to find a shortest path between the origin and destination.
Shortest path problem

In the following network various costs are assigned for the path from one node to another. For example, the cost from node 2 to node 4 is 6. The objective function considers the cost to move from each node to another from source to destination. The constraints are broken into three groups. The constraint for the origin node says that you must leave node 1 and go to node 2 or 3. The intermediate node constraints say that if you ever come into a node you must leave that node. The destination node is similar to origin node in that you must reach to this node from one of the neighboring node.
Shortest path problem

• Consider the following directed network (for an undirected network, make each arc directed in both directions, then apply the same formulation. Note that in this case you have $X_{ij}$ and $X_{ji}$ variables). The aim is to find the shortest path from node 1 to node 7.
\[ \begin{align*} 
\text{Min} & \quad 15X_{12} + 10X_{13} + 8X_{32} + 6X_{24} + 17X_{27} + 4X_{35} + 5X_{47} + 4X_{45} + 2X_{56} + 6X_{67} \\
\text{subject to} & \quad X_{12} + X_{13} = 1 \\
& \quad X_{12} + X_{32} - X_{24} - X_{27} = 0 \\
& \quad X_{13} - X_{32} - X_{35} = 0 \\
& \quad X_{24} - X_{47} - X_{45} = 0 \\
& \quad X_{35} + X_{45} - X_{56} = 0 \\
& \quad X_{56} - X_{67} = 0 \\
& \quad X_{27} + X_{47} + X_{67} = 1 \\
\end{align*} \]
Shortest path problem (answer)

• Go from 1 to 3
  Go from 3 to 5
  Go from 5 to 6
  Go from 6 to 7
• This is the shortest path with total of 22 length (cost) units.
Maximum flow Problem

• In a network with flow capacities on the arcs, the problem is to determine the maximum possible flow from the source to the sink while honoring the arc flow capacities. Consider a network with m nodes and n arcs with a single commodity flow. Denote the flow along arc (i to j) by $X_{ij}$. We associate with each arc a flow capacity, $k_{ij}$. In such a network, we wish to find the maximum total flow in the network, $F$, from node 1 to node m.
Maximum flow Problem

- In the LP formulation, the objective is to maximize $F$. The amount that leaves the origin by various routes. For every intermediate node, what comes in must be equal to what goes out. In some routes the flow can go both ways. The capacity amount that can be sent in a particular direction is also shown on the each route.
Maximum flow Problem

subject to:

Origin

Intermediate Nodes

Destination

\[
\begin{align*}
X_{12} + X_{13} - F &= 0 \\
X_{12} + X_{32} - X_{23} - X_{26} - X_{24} &= 0 \\
X_{13} + X_{23} + X_{63} - X_{32} - X_{36} - X_{35} &= 0 \\
X_{24} + X_{64} - X_{47} - X_{46} &= 0 \\
X_{35} + X_{65} - X_{56} - X_{57} &= 0 \\
X_{26} + X_{46} + X_{36} + X_{56} - X_{65} - X_{63} - X_{64} - X_{67} &= 0 \\
X_{47} + X_{57} + X_{67} - F &= 0 \\
X_{12} &\leq 10 \\
X_{13} &\leq 10 \\
X_{23} &\leq 1 \\
X_{32} &\leq 1 \\
X_{36} &\leq 6 \\
X_{36} &\leq 4 \\
X_{63} &\leq 4 \\
X_{24} &\leq 8 \\
X_{ij} &\geq 0
\end{align*}
\]

\[
\begin{align*}
X_{64} &\leq 3 \\
X_{46} &\leq 3 \\
X_{35} &\leq 12 \\
X_{65} &\leq 2 \\
X_{56} &\leq 2 \\
X_{57} &\leq 8 \\
X_{47} &\leq 7 \\
X_{67} &\leq 2
\end{align*}
\]
Maximum flow Problem (answer)

- Send 10 units from 1 to 2
  Send 7 units from 1 to 3
  Send 3 units from 2 to 6
  Send 7 units from 2 to 4
  Send 4 units from 3 to 6
  Send 6 units from 3 to 5
  Send 7 units from 4 to 7
  Send 8 units from 5 to 7
  Send 3 units from 6 to 3
  Send 2 units from 6 to 5
  Send 2 units from 6 to 7

- The maximum flow is $F = 17$ units.
critical Path Problem

• The successful management of large projects, be they construction, transportation, or financial, relies on careful scheduling and coordinating of various tasks. Critical Path Method (CPM) attempts to analyze project scheduling. This allows for better control and evaluation of the project. For example, we want to know how long will the project take? When will we be able to start a particular task? If this task is not completed on time, will the entire project be delayed? Which tasks should we speed up (crash) in order to finish the project earlier?
critical Path Problem

• Given a network of activities, the first problem of interest is to determine the length of time required to complete the project and the set of critical activities that control the project completion time. Suppose that in a given project activity network there are m nodes, n arcs (i.e. activities) and an estimated duration time, Cij, associated with each arc (i to j) in the network. The beginning node of an arc corresponds to the start of the associated activity and the end node to the completion of an activity. To find the Critical Path (CP), define the binary variables Xij, where Xij = 1 if the activity i j is on the CP and Xij = 0 otherwise. The length of the path is the sum of the duration of the activities on the path. The length of the longest path is the shortest time needed to complete the project. Formally, the CP problem is to find the longest path from node 1 to node m.
critical Path Problem

- Each arc has two roles: it represents an activity and it defines the precedence relationships among the activities. Sometimes it is necessary to add arcs that only represent precedence relationships. These dummy arcs are represented by dashed arrows. In our example, the arc from 2 to 3 represents a dummy activity.

- The first constraint says that the project must start. For each intermediate node, if we ever reach it we have to leave that node. Finally, the last constraint enforces the completion of the project.
critical Path Problem

\[
\begin{align*}
\text{max} & \quad 9X_{12} + 6X_{13} + 8X_{35} + 7X_{34} + 10X_{45} + 12X_{56} \\
\text{subject to:} & \\
\text{starting node} & \quad X_{12} + X_{13} = 1 \\
\text{intermediate nodes} & \quad X_{12} - X_{23} = 0 \\
& \quad X_{13} + X_{23} - X_{34} - X_{35} = 0 \\
& \quad X_{34} - X_{45} = 0 \\
& \quad X_{35} + X_{45} - X_{56} = 0 \\
\text{finish node} & \quad x_{56} = 1 \\
& \quad x_{ij} \geq 0
\end{align*}
\]
critical Path Problem (answer)

• Running the LP formulation on any LP solver, the critical path is:
  • From node 1 to 2
    From node 2 to 3
    From node 3 to 4
    From node 4 to 5
    From node 5 to 6
• The duration of the project is, therefore 38 time units.