Precalculus Review
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Critical Points

• For most functions, there are regions in the domain where the function is increasing (moving upward on a plot) and decreasing (moving downward).

• Wherever the function changes from increasing to decreasing and vice versa is considered a critical point of the function.

• To find the critical points, we find the first derivative of the function, and set it equal to zero. We then solve for $x$.

• If only looking at the graph of $f(x)$, look for wherever the slope of $f(x)$ changes sign.
Critical Points

• For example:

  • \( f(x) = x^2 - 4x + 3 \)

• To find critical points:

  • \( f'(x) = 2x - 4 = 0 \)

  • \( x = 2 \)

• The slope of \( f(x) = x^2 - 4x + 3 \) changes signs at \( x = 2 \)
Critical Points

- Using the graph of $f(x) = x^2 - 4x + 3$:

- We see that the slope of $f(x)$ changes at $x = 2$, where the slope changes from negative to positive.
Intervals of Slope

• After finding the critical point(s), the intervals where f(x) increases and decreases can be determined.

• We determine all parts of the general domain (-∞, ∞) where x can exist, and split up the domain into intervals based on the critical points.

• In any interval where the value for f’(x) is positive, f(x) is increasing in that interval.

• In any interval where the value for f’(x) is negative, f(x) is decreasing in that interval.
Intervals of Slope

• For example:

  • \( f(x) = 2x^3 + 6x^2 + 6x + 2 \)

• We take the derivative:

  • \( f'(x) = 6x^2 + 12x + 6 = 0 \)

• We find the zeros:

  • \( 6(x + 1)^2 = 0 \Rightarrow x = -1 \)
Intervals of Slope

• Continuing example:

• We see that there is no value of $x$ that makes $f(x)$ undefined, so $f(x)$ can be evaluated at any point. Because the only zero is $x = -1$, the two intervals are $(-\infty, -1]$ and $[-1, \infty)$. -1 is bracketed because it is an included point (does not make $f(x)$ undefined).

• We put the domain on a number line for visual representation.
Intervals of Slope

• Continuing example:

• We now plug in numbers from each interval into the f’(x) equation to determine positive and negative sections:

  • At $x = -2 \ (-\infty < -2 < -1)$, $f’(x) = 6$
  • At $x = 0 \ (-1 < 0 < \infty)$, $f’(x) = 6$

• So we see that $f’(x)$ is positive at all points except -1, where $f’(x)$ is zero. This translates to $f(x)$ increasing at all points except $x = -1$, where the slope is zero.
Intervals of Slope

• Now we can show the slopes on the number line:

-∞ + -1 + ∞

• We notice that there are no regions where the slope f(x) is negative.
Maxima and Minima

• When \( f(x) \) has intervals of positive slopes AND negative slopes, maxima and/or minima occur.

• The sign of the slope HAS TO CHANGE in order for a maximum or minimum to occur. The previous example has no such change, and so there is no maximum or minimum present in \( f(x) \).

• When the sign of the slope changes from positive to negative, there is a maximum at that point.

• When the sign of the slope changes from negative to positive, there is a minimum at that point.
Maxima and Minima

• For example:
  • \( f(x) = 2x^3 + 3x^2 - 12x + 4 \)

• We take the derivative and find the critical point(s).
  • \( f'(x) = 6x^2 + 6x - 12 = 0 \) \( \rightarrow \) \( f'(x) = 6(x - 1)(x + 2) = 0 \)

• The critical points are \( x = -2, 1 \)

• We see that there are no numbers in the domain that will cause \( f(x) \) to be undefined.
Maxima and Minima

• Continuing example:
• We construct a number line for visual representation.

• We plug in x values from within each interval into the f’(x) equation.

• At x = -3 (-∞ < -3 < -2), f’(x) = 3
• At x = 0 (-2 < 0 < 1), f’(x) = -6
• At x = 2 (1 < 2 < ∞), f’(x) = 18
Maxima and Minima

• Continuing example:
• Now we can show the slopes on the number line:

\[ \begin{array}{cccc}
-\infty & -2 & 1 & \infty \\
+ & - & + & \\
\end{array} \]

• We see that \( f'(x) \) is positive in the interval \((-\infty, -2]\), negative in \([-2, 1]\) and positive again in \([1, \infty)\).

• This means that \( f(x) \) is increasing in the interval \((-\infty, -2]\), decreasing in \([-2, 1]\) and increasing again in \([1, \infty)\).
Maxima and Minima

- We see that $f'(x)$ changes sign two times in the absolute domain $(-\infty, \infty)$; $f'(x)$ changes from positive to negative at $x = -2$, and changes from negative to positive at $x = 1$.

- This means that there is a maximum at $x = -2$, and there is a minimum at $x = 1$, as shown in the plot of $f(x)$. 

![Plot of f(x)](image)
Concavity

• Concavity describes the shape of a graph.
• When a graph opens upward, it is considered concave up:

![Concave Up Graph](image1)

• When a graph opens downward, it is considered concave down:

![Concave Down Graph](image2)
Inflection Points

• Most times, a function f(x) will have some regions that are concave upward and concave downward; the point(s) at which the concavity changes are called inflection points.

• The inflection points are the locations where the slope of the slope of f(x) (the slope of f’(x)) changes sign.

• We find the second derivative of the function f(x), and set it equal to zero. We then solve for x.
Inflection Points

• For example:
• (Continuing from previous example)
• \( f(x) = 2x^3 + 3x^2 - 12x + 4 \)
• \( f'(x) = 6x^2 + 6x - 12 = 0 \rightarrow f'(x) = 6(x - 1)(x + 2) = 0 \)

• Taking the second derivative:
• \( f''(x) = 12x + 6 = 0 \rightarrow x = -\frac{1}{2} \)

• This means that the function \( f(x) \) changes concavity at \( x = -\frac{1}{2} \)
Summary

• After viewing this tutorial, you should be confident in:

  • Identifying the locations where any function $f(x)$ is either increasing and/or decreasing by finding the critical points.

  • Determining whether or not any function $f(x)$ has any maxima and/or minima in its shape.

  • Identifying the regions of concavity of any function $f(x)$, and determining where in $f(x)$ the concavity changes (if it changes) by finding the inflection points.
References