Partial Faction Decomposition

Academic Resource Center
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What is Partial Fraction Decomposition

• This technique should be used to take a single fraction and separate it into the sum of simpler fractions
• This is can be used to simplify a term for taking the derivative
• This also works to simplify a term for integration
Finding the Partial Fraction Decomposition

Consider a rational function

\[ f(x) = \frac{P(x)}{Q(x)} \]

where \( P \) and \( Q \) are polynomials. It’s possible to express \( f \) as a sum of simpler fractions, assuming \( P \) has a smaller degree than \( Q \). If \( P \) has a larger degree than \( Q \) we must first divide \( P \) by \( Q \) so as to get a remainder \( R(x) \), which would produce the following equation:

\[ f(x) = S(x) + \frac{R(x)}{Q(x)} \]
Example 1

Simplify \( \frac{x^3 + x}{x - 1} \)

*Since the top has a higher power than the bottom we must do long division first*

\[
x - 1 \bigg| x^3 + 0 \times x^2 + x + 0
\]

\[
x^3 - x^2
\]

\[
x^2 + x
\]

\[
x^2 - x
\]

\[
2x + 0
\]

\[
2x - 2
\]

\[
\frac{x^3 + x}{x - 1} = x^2 + x + 2 + \frac{2}{x - 1}
\]
Finding the Partial Fraction Decomposition

The next step in partial fraction decomposition is to factor $Q(x)$ as much as possible. Then express $R(x)/Q(x)$ as a sum of partial fractions of the form

\[
\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}
\]

These steps can be broken down into 4 cases.
Case I: Q(x) is a product of distinct linear factors

This means that Q(x) can be rewritten as the following

\[ Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k) \]

This means that \( \frac{R(x)}{Q(x)} \) can be rewritten as the following

\[ \frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k} \]
Example 2

Simplify \( \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \)

First we must factor Q(x)

\[
2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)
\]

Now our factor can be rewritten as follows

\[
\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}
\]
Example 2 cont.

To determine the values of A, B, and C, we multiply both sides by $x(2x-1)(x+2)$ which results in the following

$$x^2 + 2x - 1 = A(2x - 1) + Bx(x + 2) + Cx(2x - 1)$$

After expanding and rewriting the equation we obtain

$$x^2 + 2x - 1 = (2A + B - C)x^2 + (3A + 2B - C)x - 2A$$
Example 2 cont.

Since the polynomials are identical, there coefficients must be equal. This gives us the following system of equations

\[2A + B + 2C = 1\]
\[3A + 2B - C = 2\]
\[-2A = -1\]

Solving this we get the following equation for our original function

\[
\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{1}{2} + \frac{1}{5} \frac{1}{2x - 1} - \frac{1}{10} \frac{1}{x + 2}
\]
Case II: $Q(x)$ is a product of linear factors, some of which repeat

Suppose that $Q(x)$ is the product of one linear factor, $r$ times. This means that $Q(x)$ can be rewritten as the following

$$Q(x) = (a_1x + b_1)^r$$

This means that $R(x)/Q(x)$ can be rewritten as the following

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$
Example 3

Simplify \( \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \)

The first step is long division resulting in the following

\[
\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}
\]

The next step is to factor \( Q(x) \) giving us the following

\[
x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1)
\]
Example 3 cont.

Now our \( \frac{R(x)}{Q(x)} \) can be rewritten as follows

\[
\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}
\]

Next we solve for \( A, B, \) and \( C \) and get our solution

\[
4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2
\]

\[
= (A + C)x^2 + (B - 2C)x + (-A + B + C)
\]

\[
A + C = 0 \quad A = 1
\]

\[
B - 2C = 4 \quad B = 2
\]

\[
-A + B + C = 0 \quad C = -1
\]

\[
\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{1}{x - 1} + \frac{2}{(x - 1)^2} - \frac{1}{x + 1}
\]
Case III: $Q(x)$ contains irreducible quadratic factors, which do not repeat.

Suppose that $Q(x)$ has a factor $ax^2+bx+c$ which cannot be factored, then, in addition to the partial factors, $R(x)/Q(x)$ will have a term of the form

\[
\frac{Ax + B}{ax^2 + bx + c}
\]
Example 4

Simplify \( \frac{2x^2 - x + 4}{x^3 + 4x} \)

Since \( x^3 + 4x = x(x^2 + x) \) cannot be factored, we write

\[
\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}
\]

By multiplying both sides by \( x(x^2 + 4) \) we get

\[
2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x
\]

\[
= (A + B)x^2 + Cx + 4A
\]
Example 4 cont.

Now solving for A, B, and C we get

\[ A + B = 2 \quad C = -1 \quad 4A = A \]
\[ A = 1, B = 1, \text{and} \quad C = -1 \]

So we see that our function reduces to this

\[
\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}
\]
Case IV: Q(x) contains a repeated irreducible quadratic factor

Suppose that Q(x) has a factor \(ax^2+bx+c\) which cannot be factored and is repeated \(r\) times, then, in addition to the partial factors, \(R(x)/Q(x)\) will have a terms of the form

\[
\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}
\]
Example 5

Simplify \( \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} \)

The form of the partial fraction decomposition is

\[
\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}
\]

By multiplying both sides by \( x(x^2 + 1)^2 \) we get

\[
-x^3 + 2x^2 - x + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x
\]

\[
= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex
\]

\[
= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A
\]
Example 5 cont.

Now solving for A, B, C, D, and E we get

\[ A + B = 0 \quad C = -1 \quad 2A + B + D = 2 \quad C + E = -1 \quad A = 1 \]
\[ A = 1, B = -1, C = -1, D = 1, \text{ and } E = 0 \]

So we see that our function reduces to this

\[
\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2}
\]
Exercises

Find the partial fractions for the following but not the coefficients

1. \( \frac{1}{x^3 + 2x^2 + x} \)

2. \( \frac{x^4 + 1}{x^5 + 4x^3} \)

3. \( \frac{t^4 + t^2 + 1}{(t^2 + 1)(t^2 + 4)^2} \)
Answers

1. \[ \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \]

2. \[ \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4} \]

3. \[ \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4} + \frac{Et+F}{(t^2+4)^2} \]
Integration with Partial Fractions

Partial fraction decomposition is useful in integration when the term that you are integrating is a function such that can be written as the division of two polynomials.
Example 6

Let's try to integrate the function in Example 3

\[
\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx = \int \left( \frac{1}{2x} + \frac{1}{5(2x - 1)} - \frac{1}{10(x + 2)} \right) \, dx
\]

\[
= \frac{1}{2} \int \frac{1}{x} \, dx + \frac{1}{5} \int \frac{1}{2x - 1} \, dx - \frac{1}{10} \int \frac{1}{x + 2} \, dx
\]

\[
= \frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x - 1| - \frac{1}{10} \ln|x + 2| + C
\]
Exercises

1. \( \int_{2}^{3} \frac{1}{x^2 - 1} \, dx \)

2. \( \int_{1}^{2} \frac{4y^2 - 7y - 12}{y(y+2)(y-3)} \, dy \)

3. \( \int \frac{10}{(x-1)(x^2 + 9)} \, dx \)
1. \( \frac{1}{2} \ln\left(\frac{3}{2}\right) \)

2. \( \frac{27}{5} \ln(2) - \frac{9}{5} \ln(3) \)

3. \( \ln|x - 1| - \frac{1}{2} \ln(x^2 + 9) - \frac{1}{3} \tan^{-1}(x/3) + C \)
Resources

• Calculus, Chapter 8.4
• http://mathworld.wolfram.com/PartialFractionDecomposition.html