Iterated Integrals

Academic Resource Center
In This Presentation...

• We will give a definition
• Explain Fubini’s Theorem
• Prove Fubini’s Theorem
• Do example problems
Definition

- In calculus, an iterated integral is the result of applying integrals to a function of more than one variable (for example $f(x,y)$ or $f(x,y,z)$) in a way that each of the integrals considers some of the variables as given constants.
Definition

- Suppose that $f$ is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y) \, dy$ to mean that $x$ is held fixed and $f(x, y)$ is integrated with respect to $y$ from $y = c$ to $y = d$. This procedure is called partial integration with respect to $y$. (Notice its similarity to partial differentiation.)
Definition

Now \( \int_{c}^{d} f(x, y) \, dy \) is a number that depends on the value of \( x \), so it defines a function of \( x \):

If we now integrate the function \( A(x) \) with respect to \( x \) from \( x = a \) to \( x = b \), we get

\[
A(x) = \int_{c}^{d} f(x, y) \, dy
\]

\[
\int_{a}^{b} A(x) \, dx = \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx
\]
Definition

- The integral on the right side of the previous equation is called an **iterated integral**. Usually the brackets are omitted. Thus means that we first integrate with respect to y from c to d and then with respect to x from a to b.

\[
\int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx
\]
Definition

• Similarly, the iterated integral

$$\int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy$$

means that from $x = a$ to $x = b$ and then we integrate the resulting function of $y$ with respect to $y$ from $y = c$ to $y = d$. 
Bubini’s Theorem

• If \( f \) is continuous on the rectangle

\[ R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}, \]

then

\[ \iint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy. \]

• More generally, this is true if we assume that \( f \) is bounded on \( R \), \( f \) is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.
Proof of Fubini’s Theorem

• The proof of Fubini’s Theorem is too difficult to include in this workshop, but we can at least give an intuitive indication of why it is true for the case where

\[ f(x, y) \geq 0. \]
Proof of Fubini’s Theorem

• Recall that if $f$ is positive, then we can interpret the double integral $\iint_R f(x, y)\,dA$ the volume $V$ of the solid $S$ that lies above $R$ and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 6, namely,

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the area of a cross-section of $S$ in the plane through $x$ perpendicular to the $x$-axis.
Proof of Fubini’s Theorem

• From Figure 1 you can see that \( A(x) \) is the area under the curve \( C \) whose equation is \( z = f(x,y) \), where \( x \) is held constant and \( c \leq y \leq d \).
Proof of Fubini’s Theorem

• Therefore

\[ A(x) = \int_c^d f(x, y) \, dy \]

and we have

\[ \iint_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx \]

• A similar argument, using cross-sections perpendicular to the y-axis as in Figure 2, shows that

\[ \iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy \]
Examples

- Evaluate the iterated integrals.

(a) \( \int_0^3 \int_1^2 x^2 y \, dy \, dx \)

(b) \( \int_1^2 \int_0^3 x^2 y \, dx \, dy \)

(a) Regarding \( x \) as a constant, we obtain

\[
\int_1^2 x^2 y \, dy = \left[ x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left( \frac{2^2}{2} \right) - x^2 \left( \frac{1^2}{2} \right) = \frac{3}{2} x^2
\]
Examples

• Thus the function $A$ in the preceding discussion is given by
  in this example. We now integrate this function of $x$ from 0 to
  3:

  $$A(x) = \frac{3}{2} x^2$$

  $$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[ \int_1^2 x^2 y \, dy \right] \, dx$$
  $$= \int_0^3 \frac{3}{2} x^2 \, dx = \frac{x^3}{2} \Bigg|_0^3 = \frac{27}{2}$$
Examples

(b) Here we first integrate with respect to:

\[
\int_{1}^{2} \int_{0}^{3} x^2 y \, dx \, dy = \int_{1}^{2} \left[ \int_{0}^{3} x^2 y \, dx \right] \, dy = \int_{1}^{2} \left[ \frac{x^3}{3} y \right]_{x=0}^{x=3} \, dy
\]

\[
= \int_{1}^{2} 9y \, dy = 9 \left[ \frac{y^2}{2} \right]_{1}^{2} = \frac{27}{2}
\]
Examples

• Evaluate the double integral \( \iint_R (x - 3y^2) \, dA \), where \( R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\} \).

SOLUTION 1 Fubini’s Theorem gives

\[
\iint_R (x - 3y^2) \, dA = \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx = \int_0^2 [xy - y^3]_{y=1}^{y=2} \, dx
\]

\[
= \int_0^2 (x - 7) \, dx = \left[ \frac{x^2}{2} - 7x \right]_0^2 = -12
\]
Examples

• SOLUTION 2 Again applying Fubini’s Theorem, but this time integrating with respect to x first, we have

\[
\iint_R (x - 3y^2) \, dA = \int_1^2 \int_0^2 (x - 3y^2) \, dx \, dy
\]

\[
= \int_1^2 \left[ \frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} \, dy
\]

\[
= \int_1^2 (2 - 6y^2) \, dy = 2y - 2y^3 \bigg|_1^2 = -12
\]
Examples

• Evaluate \( \iint_{R} y \sin(xy) \, dA \), \( R = [1, 2] \times [0, \pi] \).

SOLUTION 1: If we first integrate with respect to \( x \), we get

\[
\iint_{R} y \sin(xy) \, dA = \int_{0}^{\pi} \int_{1}^{2} y \sin(xy) \, dx \, dy = \int_{0}^{\pi} \left[ -\cos(xy) \right]_{x=1}^{x=2} \, dy
\]

\[
= \int_{0}^{\pi} (-\cos 2y + \cos y) \, dy
\]

\[
= -\frac{1}{2} \sin 2y + \sin y \bigg|_{0}^{\pi} = 0
\]
Examples

SOLUTION 2 If we reverse the order of integration, we get

\[ \iint_R y \sin(xy) \, dA = \int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx \]

To evaluate the inner integral, we use integration by parts with

\[
\begin{align*}
u &= y & \quad dv &= \sin(xy) \, dy \\
\frac{du}{dx} &= dy & \quad v &= -\frac{\cos(xy)}{x}
\end{align*}
\]
Examples

and so

\[
\int_0^\pi y \sin(xy) \, dy = -\frac{y \cos(xy)}{x} \bigg|_{y=0}^{y=\pi} + \frac{1}{x} \int_0^\pi \cos(xy) \, dy \\
= -\frac{\pi \cos \pi x}{x} + \frac{1}{x^2} \left[ \sin(xy) \right]_{y=0}^{y=\pi} \\
= -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}.
\]

If we now \( u = -1/x \) and \( dv = \pi \cos \pi x \, dx \), we get

\[
\int \left( -\frac{\pi \cos \pi x}{x} \right) \, dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2} \, dx
\]

and
Examples

Therefore

\[ \int \left( -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \right) \, dx = -\frac{\sin \pi x}{x} \]

and so

\[ \int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx = \left[ -\frac{\sin \pi x}{x} \right]_1^2 \]

\[ = -\frac{\sin 2\pi}{2} + \sin \pi = 0 \]
Thanks!
References

• Calculus – Stewart 6th Edition
  • Section 15.2 “Iterated Integrals”
• http://en.wikipedia.org/wiki/Iterated_integral
• Prepared by Shenghze Jin