Equations of Lines & Planes

Academic Resource Center
Definition

A Line in the space is determined by a point and a direction.

Consider an line L and a point \( P(x_0,y_0,z_0) \) on L. Direction of this line is determined by a vector \( v \) that is parallel to Line L.

Let \( P(x,y,z) \) be any point on the Line
Let \( r_0 \rightarrow \) is the Position vector of point \( P_0 \)
\( r \rightarrow \) is the Position vector of point \( P \)
Definition

Then **vector equation of line** is given by

\[ r = r_0 + vt \]

Where \( t \) is a scalar

Let \( v = <a, b, c> \)

\[ r = <x_0, y_0, z_0> \]

Hence the **parametric equation of a line** is:-
Parametric Equations of a line

\[ \langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \]

Hence we get,

\[ x = x_0 + at \]
\[ Y = y_0 + bt \]
\[ Z = z_0 + ct \]
Symmetric Equations of Line

- Symmetric Equations of line is given by:

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]
Example 1

**Question:** - Find the vector equation and Parametric equations of the line through (1,2,3) and parallel to vector 3i+2j-k

**Solution:** - Given \( \mathbf{v} = i+2j+3k \)

\[
r_0 = \langle 1, 2, 3 \rangle
\]

\( = i + 2j + 3k \)

Hence the **vector equation of line** is

\[
r = i + 2j + 3k + t(3i + 2j - k)
\]

\( = (1 + 3t)i + (2 + 2t)j + (3 - t)k \)
Example 1 (Continued)

Since \( r = xi + yj + zk \)

Hence we get **parametric equation of line** is:

- \( x = 1 + 3t \)
- \( y = 2 + 2t \)
- \( z = 3 - t \)
Example 2

Question: - Find the symmetric equation for line through point (1, -5, 6) and is parallel to vector <-1, 2, -3>

Solution: - We know that symmetric equation of line is given by:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Here

$$a = -1, b = 2, c = -3$$
Example 2 (Continued)

• Hence we get

\[
\frac{x - 1}{-1} = \frac{y + 5}{2} = \frac{z - 6}{-3}
\]

Hence the result
Planes

• The plane in the space is determined by a point and a vector that is perpendicular to plane.

• Let \( P(x_0, y_0, z_0) \) be given point and \( n \) is the orthogonal vector.

• Let \( P(x, y, z) \) be any point in space and \( r, r_0 \) is the position vector of point \( P \) and \( P_0 \) respectively.

• Then vector equation of plane is given by:-
Planes

- \( n \cdot r = n \cdot r_0 \)
- Let \( n = <a, b, c> \)
- \( r = <x, y, z> \)
- \( r_0 = <x_0, y_0, z_0> \)
- Hence the vector equation becomes:
  - \( <a, b, c> \cdot <x, y, z> = <a, b, c> \cdot <x_0, y_0, z_0> \)
- Hence the scalar equation of plane is given by
Planes

- $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$
- We can also write above equation of plane as:
  - $ax+by+cz+d=0$
- Where $d=-(ax_0+by_0+cz_0)$
Example 3

**Question:** Find the equation of plane through point (1,-1,1) and with normal vector i+j-k

**Solution:** Given point is (1,-1,1)

Here a=1, b=1, c=-1

We know that equation of plane is given by:

\[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

\[ 1(x-1) + 1(y+1) - 1(z-1) = 0 \]

\[ x+y-z+1=0 \]
Example 4

• **Question**: Find the line of intersection of two planes $x+y+z=1$ and $x+2y+2z=1$

• **Solution**: Let $L$ is the line of intersection of two planes.

We can find the point where Line $L$ intersects $xy$ plane by setting $z=0$ in above two equations, we get:

- $x+y=1$
- $x+2y=1$
Example 4 (Continued)

- By solving for $x$ and $y$ we get,
  - $x=1$
  - $y=0$
- Hence the Point on Line $L$ is $(1,0,0)$
- Line $L$ lies in both planes so it is perpendicular to both normal vectors
- $a=i+j+k$ and $b=i+2j+2k$
- Hence vector $v$ which is given by
- $v=\mathbf{a} \times \mathbf{b}$ is parallel to $L$
Example 4(Continued)

• Hence
• $v=0i-j+k$
• We know that equation of line is given by
  
  \begin{align*}
  x &= x_0 + at \\
  y &= y_0 + bt \\
  z &= z_0 + ct
  \end{align*}

  Hence we get,
• $x=1, y=-t, z=2t$
Example 5

• **Question:** Find the equation of plane through the points $(0,1,1),(1,0,1)$ and $(1,1,0)$

• **Solution:** Let $p (0,1,1)$, $q(1,0,1)$ and $r(1,1,0)$ denote the given points.

Let $a=<1-0,0-1,1-1>=<1,-1,0>$

And $b=<1-1,1-0,0-1>=<0,1,-1>$

$n=\mathbf{a} \times \mathbf{b}$ is the orthogonal vector of the plane.

Hence $n=i+j+k$
Example 5 (Continued)

• Hence the equation of plane through (0,1,1) and perpendicular to vector \( \mathbf{i} + \mathbf{j} + \mathbf{k} \) is:

  
  \[ 1(x-0) + 1(y-1) + 1(z-1) = 0 \]

  
  \[ x + y - 1 + z - 1 = 0 \]

  
  \[ x + y + z = 2 \]

  
  • Hence the result
Practice Problems

1) Find the parametric equation of line through point (1, -1, 1) and parallel to line $x + 2 = y/2 = z - 3$.

2) Find the equation of plane through points (3, -1, 2), (8, 2, 4) and (-1, -2, -3).

3) Find the symmetric equation of line of intersection of planes.
Practice Problems

5x-2y-2z=1,
4x+y+z=6

• 4) Find the point at which line
  x=3-t, y=2+t, z=5t
  Intersects the plane x-y+2z=9
Answer To Practice Problems

1) \( x = 1 + t, \)  
\( y = -1 + 2t, \)  
\( z = 1 + t \)

2) \(-13x + 17y + 7z = -42\)

3) \(x = 1, y - 2 = -z\)

4) Point is (2,3,5)
Important Tips for Practice Problem

• For **Question 1**, direction number of required line is given by (1,2,1), since two parallel lines have same direction numbers.

• For **question 2**, see solved example 5

• For **question 3**, see solved example 4

• **For Question 4**, put the value of $x, y, z$ in the equation of plane and then solve for $t$. After getting value of $t$, put in the equations of line you get the required point.