Solution of Bernoulli’s Equations

Academic Resource Center
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First Order differential equations

- A differential equation having a first derivative as the highest derivative is a first order differential equation.
- If the derivative is a simple derivative, as opposed to a partial derivative, then the equation is referred to as ordinary.
Linearity of Differential Equations

• The terminology ‘linear’ derives from the description of a line.
• A line, in its most general form, is written as $Ax + By = C$
• Similarly, if a differential equation is written as $f(x) \frac{dy}{dx} + g(x)y = h(x)$
• Then this equation is termed linear, as the highest power of $\frac{dy}{dx}$ and y is 1. They are analogous to x and y in the equation of a line, hence the term linear.
Typical form of Bernoulli’s equation

- The Bernoulli equation is a Non-Linear differential equation of the form
  \[ \frac{dy}{dx} + P(x)y = f(x)y^n \]
- Here, we can see that since \( y \) is raised to some power \( n \) where \( n \neq 1 \).
- This equation cannot be solved by any other method like homogeneity, separation of variables or linearity.
Examples

- Here are a few examples of Bernoulli’s Equation

\[
x \frac{dy}{dx} + y = x^2 y^2
\]

\[
y^2 x \frac{dy}{dx} = xy^3 + 1
\]

\[
t^2 \frac{dy}{dt} + y^2 = ty
\]
Method of Solution

• The first step to solving the given DE is to reduce it to the standard form of the Bernoulli’s DE. So, divide out the whole expression to get the coefficient of the derivative to be 1.
• Then we make a substitution \( u = y^{1-n} \)
• This substitution is central to this method as it reduces a non-linear equation to a linear equation.
Bernoulli Substitution

• So if we have $u = y^{1-n}$,

  then $\frac{du}{dx} = (1 - n)y^{-n} \frac{dy}{dx}$

• From this, replace all the $y$’s in the equation in terms of $u$ and replace $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$ and $u$.

• This will reduce the whole equation to a linear differential equation.
Example Problem

Let us try to solve the given differential equation

\[ x^2 \frac{dy}{dx} - y^2 = 2xy \]

First of all, we rearrange it so that the derivative and first power of \( y \) are on one side.

\[ x^2 \frac{dy}{dx} - 2xy = y^2 \]

Then we divide out by \( x^2 \) to make the co-efficient of the derivative equal to 1.
Then, that leaves us with
\[
\frac{dy}{dx} - \frac{2}{x} y = \frac{1}{x^2} y^2
\]
Here, we identify the power on the variable \( y \) to be 2. Therefore, we make the substitution
\[
u = y^{1-2} = y^{-1}
\]
Thus,
\[
\frac{du}{dx} = -y^{-2} \frac{dy}{dx}
\]
Then, knowing that:

\[
\frac{du}{dx} = -y^{-2} \frac{dy}{dx} \quad \text{and} \quad u = y^{-1}
\]

We have \( y = \frac{1}{u} \) and \( -\frac{1}{u^2} \frac{du}{dx} = \frac{dy}{dx} \)

We substitute these expressions into our Bernoulli’s equation and that gives us

\[
-\frac{1}{u^2} \frac{du}{dx} - \frac{2}{x u} = \frac{1}{x^2} \frac{1}{u^2}
\]
Once we have reached this stage, we can multiply out by $-u^2$ and then the DE simplifies to

$$\frac{du}{dx} + \frac{2}{x}u = -\frac{1}{x^2}$$

This is now a linear differential equation in $u$ and can be solved by the use of the integrating factor. That’s the rational behind the use of the specialized substitution.
Proceeding, we identify \( P(x) = \frac{2}{x} \), we have the integrating factor

\[
I(x) = e^{\int \frac{2}{x} \, dx} \\
I(x) = e^{2 \ln(x)} = x^2
\]

Then multiplying through by \( I(x) \), we get

\[
x^2 \frac{du}{dx} + 2xu = -1
\]

Which simplifies to

\[
\frac{d}{dx} (x^2 u) = -1
\]
We now integrate both sides,

$$\int \frac{d}{dx} (x^2 u) = \int -1dx$$

This gives

$$x^2 u = -x + C$$

$$u = \frac{-1}{x} + \frac{C}{x^2}$$

We have now obtained the solution to the simplified DE. All we have to do now is to replace $u = \frac{1}{y}$.
Contd.

Our final answer will therefore look like

\[ y = \left( \frac{-1}{x} + \frac{C}{x^2} \right)^{-1} \]
Practice Problems

Try practicing these problems to get the hang of the method. You can also try solving the DEs given in the examples section.

1. \( x \frac{dy}{dx} - (1 + x)y = xy^2 \)

2. \( \frac{dy}{dx} = y(xy^3 - 1) \)

3. \( 3(1 + t^2) \frac{dy}{dx} = 2ty(y^3 - 1) \)
References

• A first Course in Differential Equations 9th Ed., Dennis Zill.
• Elementary Differential Equations, Martin & Reissner
• Differential and Integral Calculus Vol 2, N. Piskunov
• Workshop developed by Abhiroop Chattopadhyay