AMPERE’S LAW
Introduction

- A useful law that relates the net magnetic field along a closed loop to the electric current passing through the loop.

- First discovered by André-Marie Ampère in 1826
Definition

- The integral around a closed path of the component of the magnetic field tangent to the direction of the path equals $\mu_0$ times the current intercepted by the area within the path

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$
Or, in a simplified scalar form

$$\oint B_\parallel \cdot ds = \mu_0 I$$

Thus the line integral (circulation) of the magnetic field around some arbitrary closed curve is proportional to the total current enclosed by that curve.
Important Notes

- In order to apply Ampère’s Law all currents have to be steady (i.e. do not change with time)
- Only currents crossing the area *inside* the path are taken into account and have some contribution to the magnetic field
- Currents have to be taken with their algebraic signs (those going “out” of the surface are positive, those going “in” are negative)- use right hand’s rule to determine directions and signs
• The total magnetic circulation is zero only in the following cases:
  - the enclosed net current is zero
  - the magnetic field is normal to the selected path at any point
  - the magnetic field is zero
• Ampère’s Law can be useful when calculating magnetic fields of current distributions with a high degree of symmetry (similar to symmetrical charge distributions in the case of Gauss’ Law)
Example: Calculating Line Integrals

The figure below shows two closed paths wrapped around two conducting loops carrying currents $i_1$ and $i_2$. What is the value of the integral for (a) path 1 and (b) path 2?

![Diagram](image)

To do this you have to use the right hand rule to check whether the currents are positive or negative relative to the path. On path 1 $i_1$ penetrates in the negative direction while $i_2$ penetrates in the positive direction, so $\int \mathbf{B} \cdot ds = \mu_0 \left( i_2 - i_1 \right)$.

On path 2 $i_1$ penetrates twice in the negative direction and $i_2$ once in the negative direction so $\int \mathbf{B} \cdot ds = -\mu_0 \left( 2i_1 + i_2 \right)$.
Example: Coaxial Cable

A coaxial cable consists of a solid inner conductor of radius $a$, surrounded by a concentric cylindrical tube of inner radius $b$ and outer radius $c$. The conductors carry equal and opposite currents $I_0$ distributed uniformly across their cross-sections. Determine the magnitude and direction of the magnetic field at a distance $r$ from the axis. Make a graph of the magnitude of the magnetic field as a function of the distance $r$ from the axis.

(a) $r < a$;
(b) $a < r < b$;
(c) $b < r < c$;
(d) $r > c$. 
Solution

(a) The enclosed current is  \( I_{enc} = I_0 \left( \frac{\pi r^2}{\pi a^2} \right) = \frac{I_0 r^2}{a^2} \). Applying Ampere’s law, we have

\[
B(2\pi r) = \mu_0 \frac{I_0 r^2}{a^2} \quad \text{or} \quad B = \frac{\mu_0 I_0}{2\pi a^2} r, \quad \text{running counterclockwise when viewed from left}
\]

(b) The enclosed current is  \( I_{enc} = I_0 \). Applying Ampere’s law, we obtain

\[
B(2\pi r) = \mu_0 I_0 \quad \text{or} \quad B = \frac{\mu_0 I_0}{2\pi r}, \quad \text{running counterclockwise when viewed from left}
\]
(c) 
\[ I_{enc} = I_0 - I_0 \left( \frac{\pi r^2 - \pi b^2}{\pi c^2 - \pi b^2} \right) = \frac{I_0(c^2 - r^2)}{c^2 - b^2} \]

Applying Ampere’s law,

\[ B(2\pi r) = \mu_0 \frac{I_0(c^2 - r^2)}{c^2 - b^2} \]

or \[ B = \frac{\mu_0 I_0(c^2 - r^2)}{2\pi(c^2 - b^2)r} \], running counterclockwise when viewed from left

(d) 
\[ B = 0 \text{ since } I_{enc} = 0 \]
Example: Cylindrical Conductor

Consider an infinitely long, cylindrical conductor of radius $R$ carrying a current $I$ with a non-uniform current density $J = \alpha r^2$, where $\alpha$ is a constant and $r$ is the distance from the center of the cylinder.

(a) Find the magnetic field everywhere.

(b) Plot the magnitude of the magnetic field as a function of $r$. 
Solution

(a) The enclosed current is given by

\[ I_{\text{enc}} = \int \vec{J} \cdot d\vec{A} = \int (\alpha r'^2) (2\pi r' dr') = \int 2\pi \alpha r'^3 dr' \]

For \( r < R \),

\[ I_{\text{enc}} = \int_{0}^{r} 2\pi \alpha r'^3 \, dr' = \frac{\pi \alpha r^4}{2} \]

Applying Ampere’s law, the magnetic field is given by

\[ B(2\pi r) = \frac{\mu_0 \pi \alpha r^4}{2} \]

or

\[ B = \frac{\mu_0 \alpha}{4} r^3 \]
For $r > R$,

$$I_{enc} = \int_{0}^{R} 2\pi \alpha r^3 \, dr' = \frac{\pi \alpha R^4}{2}$$

Applying Ampere’s law, the magnetic field is given by

$$B(2\pi r) = \frac{\mu_0 \pi \alpha R^4}{2}$$

or

$$B = \frac{\mu_0 \alpha R^4}{4r}$$
(b) Plot the magnitude of the magnetic field as a function of $r$. 

\[ B \propto \frac{\mu_0 \alpha R^3}{4} \]

\[ B \propto r^3 \]

\[ B \propto \frac{1}{r} \]
Example: Two Long Solenoids

Two long solenoids are nested on the same axis, as in the figure below. The inner solenoid has radius $R_1$ and $n_1$ turns per unit length. The outer solenoid has radius $R_2$ and $n_2$ turns per unit length. Each solenoid carries the same current $I$ flowing in each turn, but in opposite directions, as indicated on the sketch.
Use Ampere’s Law to find the direction and magnitude of the magnetic field in the following regions. Be sure to show your Amperian loops and all your calculations.

a) \(0 < r < R_1\)
b) \(R_1 < r < R_2\)
c) \(R_2 < r\)
Solution

(a) \(0 < r < R_1\);

To solve for the magnetic field in this case, we take the top rectangular loop shown in the figure. The current through the loop is

\[ I_{\text{enc}} = -n_1 \ell I + n_2 \ell I = (-n_1 + n_2) \ell I \]
The loop has four segments. Along two of those (top and bottom, horizontal), $\vec{B}$ is perpendicular to $d\vec{s}$, and $\vec{B} \cdot d\vec{s} = 0$. On the other hand, along the outer vertical segment, $\vec{B} = 0$. Thus, using Ampere’s law $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}}$, we have

$$\oint \vec{B} \cdot d\vec{s} = B\ell + 0 + 0 + 0 = B\ell = \mu_0 (-n_1 \ell I + n_2 \ell I) \quad \Rightarrow \quad \vec{B} = \mu_0 I (-n_1 + n_2) \hat{k}$$
(b) \( R_1 < r < R_2 \)

To solve for the magnetic field in this case, we take the bottom rectangular loop shown in the figure. The current through the loop is

\[
I_{enc} = n_2 \ell I
\]

The loop has four segments. Along two of those (top and bottom, horizontal), \( \vec{B} \) is perpendicular to \( d\vec{s} \), and \( \vec{B} \cdot d\vec{s} = 0 \). On the other hand, along the outer vertical segment, \( \vec{B} = 0 \). Thus, using Ampere's law \( \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \), we have

\[
\oint \vec{B} \cdot d\vec{s} = B\ell + 0 + 0 + 0 = B\ell = \mu_0 n_2 \ell I \quad \Rightarrow \quad \vec{B} = \mu_0 n_2 I \hat{k}
\]
(c) \(R_2 < r\)

Since the net current enclosed by the Amperian loop is zero, the magnetic field is zero in this region.
References

• Physics for Engineers and Scientists, Chapter 29
  Hans C. Ohanian, John T. Markert

• Fundamentals of Physics, Chapter 31
  Halliday, Resnick, Walker

• http://ocw.mit.edu